Wave Propagation Along a Thin Vertical Wire Antenna Placed in a Horizontally Layered Media

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Abstract: A theoretical and numerical analysis of the wave propagation along a long thin wire antenna is presented. The wire is assumed to be placed vertically in a conductive inhomogeneous medium, which is represented by a finite set of horizontal plane layers characterized by conductivities. The current distribution along the wire antenna is obtained as the solution of the electric field integral equation that is solved by the method of moments. Numerical results are presented and compared to similar works.

Keywords: Wire antenna, layered media, moment method

1. Introduction

Wave propagation along a wire antenna placed in a conductive medium is an interesting topic in low frequency communications. In 1979, Wait and Hill [6] calculated the current distribution along a drill rod surrounded by conducting host rock. They assumed an infinite long perfect conducting rod located in a homogeneous lossy medium. The source was assumed to be a toroidal coil emitting a 5KHz electromagnetic wave riding on the rod. They found that the attenuation would be at least as great as that of plane waves in the conductive medium surrounding the rod. In 1989, DeGauque and Grudzinski [5] studied the current distribution along a drillstring of finite conductivity embedded in a homogeneous conductive medium using Pocklington's integral equation [1, Ch.8]. The source was assumed to be an electric delta gap which could be viewed as a simple electric dipole. The frequency range of the emitted signal was 0.1-10Hz. They found that the surface impedance played a major role in attenuating the signal. However, for frequencies below a few hertz, the attenuation did not vary much and the optimum frequency for maximum data rate was argued to be about 3Hz.

To the best of the author's knowledge, an analytical solution of the current distribution along a long metal wire antenna in a conductive inhomogeneous medium is not known. To address this issue, a theoretical and numerical analysis is presented. The wire antenna is assumed to be placed vertically in a conductive inhomogeneous medium, represented by a finite set of horizontal plane layers, each of which is characterized by an individual conductivity. The current distribution along the wire antenna is obtained as the solution of the electric field integral equation that is solved by the method of moments.
2. Theoretical approach

A. The field from a wire current in a homogeneous medium

A long straight wire antenna of length $l$ and radius $a$ is assumed to be placed vertically in a homogeneous medium and oriented along the positive $z$-axis. The electric field $\mathbf{E}(\mathbf{r})$ at an observation point $(\mathbf{r})=(x,y,z)$ is related to the current density $\mathbf{J}(\mathbf{r'})$ on the wire antenna by the integral [3, Eq.(7.1.2)]

$$
\mathbf{E}(\mathbf{r}) = i\omega\mu \int_{0}^{l} d^{3}\mathbf{r}' \mathbf{G}(\mathbf{r},\mathbf{r'}) \mathbf{J}(\mathbf{r'}). 
$$

(1)

The dyadic Green’s function $\mathbf{G}(\mathbf{r},\mathbf{r'})$ is defined as [3, Eq.(7.1.19)]

$$
\mathbf{G}(\mathbf{r},\mathbf{r'}) = \left[ 1 + \frac{\nabla\nabla}{k^2} \right] g(\mathbf{r},\mathbf{r'}),
$$

(2)

where $\mathbf{I}$ denotes the unit dyad and $g(\mathbf{r},\mathbf{r'})$ is the scalar Green’s function. In an unbounded homogeneous medium $g(\mathbf{r},\mathbf{r'}) = \frac{e^{i|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$ and $k = \omega\sqrt{\mu(\epsilon + i\sigma/\omega)}$ (Im($\epsilon$)>0) is the complex wave number. In the following, the antenna is assumed to be very thin compared to its length, i.e. $l>>a$. Under this assumption, the current density along the wire can be approximately written as:

$$
\mathbf{J}(\mathbf{r'}) = I(z') \delta(x') \delta(y') \hat{z},
$$

(3)

where $I(z')$ is assumed to be an equivalent line-source current [1,Ch8]. Inserting (2) and (3) into (1), the $z$-component of the electric field directed along the antenna can be written as

$$
E_z(\mathbf{r}) = \frac{i\omega\mu}{4\pi k^2} \int_{0}^{l} d\rho' I(z') \left[ k^2 + \frac{\partial^2}{\partial z^2} \right] \frac{e^{ikR}}{R}
$$

(4)

where $R = \sqrt{\rho^2 + (z-z')^2}$, and $\rho = \sqrt{x^2 + y^2}$.

B. The field from a wire current in a layered medium

The wire antenna is assumed to be placed vertically in a conductive inhomogeneous medium represented by a finite set of horizontal plane layers, where each layer is characterized by an individual conductivity. In [3,Ch.2], a method to obtain a solution for the electromagnetic fields generated by a point or line source embedded in such a multilayered profile is presented. The heart of the approach is based on the fact that nonplanar waves generated by finite sources can be expanded into an integral summation of plane waves. The mathematical identity is known as Weyl's identity [3, Eq. (2.2.27)], and it represents a plane-wave expansion of a spherical wave. Once this is done, the general theory of reflection and transmission of plane waves can be used to find the electromagnetic fields within any of the layers in response to a source within one of the layers. In this section, this method is used to find an expression for the $z$-component of the electric field generated by the antenna. Due to limited space available, a detailed explanation of the method and a presentation of all the involved symbols and expressions are not included. The interested reader is rather referred to [3,Ch.2] for further details. Note that in this paper the $z$-axis points upwards, whereas in [3,Fig. 2.4.1], it points downwards. As a result, there are some differences in the subindexes of the expressions in this paper and the similar ones in [3].

Since the antenna is assumed to be placed along the $z$-axis of a cylindrical coordinate system, the electric field component in question is the $z$-component which is directed along the antenna. To derive an expression for this field, the Sommerfeld identity [3, Eq. (2.2.30)] is used as a starting point:

$$
\frac{e^{ikR}}{R} = i\int_{0}^{\infty} dk_{\rho} k_{\rho} J_0(k_{\rho}\rho) e^{ik_{\rho}l}.
$$

(5)

which shows that a spherical wave can be expanded into an integral summation of cylindrical waves in the $\rho$ direction times a plane wave in the $z$-direction over all wave numbers $k_{\rho}$. In (5), $J_0$ denotes a...
Bessel function of the first kind and zeroth order, and \( k_z = \sqrt{k^2 - k_p^2} \) (\( \text{Im}(k_z) > 0 \)) denotes the \( z \)-component of the wave vector \( k \). In the following, it is assumed that a source at \( z=z' \) is embedded in layer \( m \) of a multilayered profile (see [3,Fig.2.4.1] or [2,Fig.1] as a reference). Since the positive \( z \)-axis in this case points upwards, layer \( m \) is mathematically constrained to be within the limits \( d_{m-1} < z' < d_m \). The \( z \) variation of an upgoing and downgoing wave can be expressed as \( F(z,z') \). The field inside layer \( m \) due to a source in layer \( m \) can be divided into two parts, representing an upgoing wave \( (F_u) \) and a downgoing wave \( (F_d) \), which can be expressed as

\[
F_m(z,z') = \begin{cases} 
  F_u(z,z') = A_u^+ e^{ik_z(z'-z)} + \tilde{R}_{m,m+1} e^{ik_z(2d_{m-1}-z-z')}, & z > z' \\
  F_d(z,z') = A_d^+ e^{-ik_z(z'-z)} + \tilde{R}_{m,m-1} e^{ik_z(2d_{m+1}-z-z')}, & z < z'
\end{cases}
\]  

(6)

where \( \tilde{R}_{m,m+1} \) and \( \tilde{R}_{m,m-1} \) represent generalized reflection coefficients for waves emanating from layer \( m \) into layer \( m+1 \) and \( m-1 \), respectively, \( d_a \) denotes the \( z \)-coordinate of the interface separating medium \( m \) and \( m+1 \), and \( k_{\text{inc}} \) is used to indicate the \( z \)-component of the wave vector \( k \) of layer \( m \). The amplitudes in (6) are equal to

\[
A_u^+ = \tilde{M}_m [1 + \tilde{R}_{m,m+1} e^{2ik_{\text{inc}}(d_{m+1}-z)}],
\]

(7)

\[
A_d^+ = \tilde{M}_m [1 + \tilde{R}_{m,m+1} e^{2ik_{\text{inc}}(z-d_{m+1})}],
\]

(8)

in which

\[
\tilde{M}_m = [1 - \tilde{R}_{m,m-1} \tilde{R}_{m,m+1} e^{2ik_{\text{inc}}(d_{m+1}-d_{m-1})}]^{-1}.
\]

(9)

In a similar fashion, the field variation in layer \( m+1 \) and \( m-1 \) can be written as

\[
F_{m+1}(z,z') = A_{m+1}^+ e^{ik_z(z'-z)} + \tilde{R}_{m+1,m+2} e^{ik_z(2d_{m+1}-z-z')},
\]

(10)

\[
F_{m-1}(z,z') = A_{m-1}^+ e^{ik_z(z'-z)} + \tilde{R}_{m-1,m-2} e^{ik_z(2d_{m-1}-z-z')},
\]

(11)

where

\[
A_{m+1}^+ = \frac{T_{m+1,m+2} e^{ik_z(d_{m+1}-z')}}{1 - R_{m+1,m+2} e^{2ik_z(d_{m+1}-d_m)}},
\]

(12)

\[
A_{m-1}^+ = \frac{T_{m-1,m} e^{ik_z(z'-d_{m+1})}}{1 - R_{m-1,m} e^{2ik_z(d_{m+1}-d_m)}},
\]

(13)

The expressions \( T_{m+1,m+2} \) and \( R_{m+1,m} \) as well as \( T_{m+1,m+2} \) and \( R_{m-1,m} \) represent transmission and reflection coefficients between layer \( m \) and the adjacent layers \( m+1 \) and \( m-1 \) only. The field variations within the layers above \( m+1 \) or below \( m-1 \) are obtained through a recursive approach. The above expressions for \( F(z,z') \) may be introduced into (5), and when combined with (4), the \( z \)-component of the electric field directed along the antenna can be expressed as

\[
E_z(a,z) = \frac{-\alpha k u}{4\pi k_z^2} \int_0^l dz' l(z')(k_z^2 + \frac{\partial^2}{\partial z'^2}) \int_0^{\infty} dk \frac{k^2}{k_z} J_0(k_{\rho} a) F(z,z').
\]

(14)

A thing to notice is that when \( z=0 \),

\[
(k_z^2 + \frac{\partial^2}{\partial z'^2}) \int_0^{\infty} dk \frac{k^2}{k_z} J_0(k_{\rho} a) F(z,z') = \int_0^{\infty} dk \frac{k^2}{k_z} J_0(k_{\rho} a) F(z,z'),
\]

(15)

goes to infinity. The reason for this is that when \( z=0 \), it indicates wave propagating in \( \rho \) direction and \( k_z = 0 \). In this case, the integrand in (4) has the following closed form solution [3, pp. 118]

\[
(k_z^2 + \frac{\partial^2}{\partial z'^2}) = \frac{e^{iKR}}{R^3} (1 - ikR)(2R^2 - 3a^2) + (kaR)^2.
\]

(16)

The righthand side of (16) represents the core of Pocklington’s integral equation [1].
C. Calculating the current by the method of moments

Equation (4) is the electric field integral equation with \( I(z') \) as the unknown current distribution and the electric field as the known excitation function. The current may be determined by the application of boundary conditions on the surface of the antenna. For a perfect conductor, the electric field along the antenna is equal to zero \( E_z(a,z) = 0 \) at all points except at the source. The integral in (4) can then be solved with respect to \( I(z') \) by using the method of moments \([1],[4]\). The length of the wire antenna is discretized into \( N \) equidistant points \( z_i = (i-1/2)\Delta z \), each of length \( \Delta z = l/N \). Under the assumption that the unknown current \( I(z) \) varies slowly over the length \( \Delta z \), and therefore can be approximated to be constant and equal to \( I(z_i) \), the following set of linear equation is obtained:

\[
G(z,z')I(z') = b(z),
\]

where the vector \( I(z') \) represents the current at the points \( (z_1', z_2', \cdots, z_N') \) along the wire antenna. The matrix \( G(z,z') \) is defined as

\[
G(z,z') = \begin{bmatrix}
g(z_1,z_1') & g(z_1,z_2') & \cdots & g(z_1,z_N') 
g(z_2,z_1') & g(z_2,z_2') & \cdots & g(z_2,z_N') 
\vdots & \vdots & \ddots & \vdots 
g(z_N,z_1') & g(z_N,z_2') & \cdots & g(z_N,z_N')
\end{bmatrix},
\]

in which the individual matrix elements are equal to

\[
g(z_i,z_j) = -\frac{\omega \mu \varepsilon}{4\pi} \int_0^{\infty} dk \rho_k \frac{k^3_\rho}{k^3_m} J_0(k_\rho a) F(z_i,z_j). \tag{19}
\]

The symbols \( \sigma_s \) and \( k_s \) represent the conductivity and wavenumber of the antenna respectively.

3. Implementation and numerical results

The numerical results in this paper are obtained using MATLAB. To solve the double integral in (14), the inner integral with respect to \( k_\rho \) needs to be solved first. However, since \( 1/k_\rho = 1/\sqrt{k^2 - k^2_\rho} \), there is a pole at \( k_\rho = k \) for \( k_\rho \in (0,\infty) \). This means that \( k_\rho \) should be selected more finely near the pole. For the integration with respect to \( z' \), both the built-in MATLAB function \( \text{quad} \) and a direct integration technique have been used. The \( \text{quad} \) function runs faster and is more stable than a direct integration approach. In Fig.1, the result of the direct integration approach is depicted for a perfect conductive wire antenna in a homogeneous medium with conductivity \( \sigma_1 = 4 \text{S/m} \), operating at a frequency of 100Hz. It shows that the results vary with the number of selected points. In Fig.2, similar results are depicted when using the \( \text{quad} \) function. It is observed that the \( \text{quad} \) function achieves the same result as the direct integration approach but with less points involved. To verify that the numerical results in this paper are consistent with previously reported results in homogeneous media, the results from \([5,\text{Fig.6}]\) and \([5,\text{Fig.4}]\) have been reproduced and are presented in Fig.3 and Fig.4, respectively. By inspection, the results obtained in this paper are in good agreement with the previously published results. However, despite a similar shape of the curves, the attenuation of the current distribution in this paper is larger than the those presented in \([5,\text{Fig.4}]\). As a check, the results using \( \text{quad} \) was compared to a similar model in CST. In Fig.5, it shows that the results in this paper are in good agreement with the result from CST.
Fig. 1. Theoretical current distribution along a perfect conductive wire antenna of length $l=100m$ and radius $a=0.1m$. The source is located at $0.1l$. The antenna operates at $f=100Hz$ and is placed in a conductive homogeneous medium with conductivity $\sigma_1=4S/m$. The result is obtained by direct integration with different filament numbers.

Fig. 2. Theoretical current distribution along a perfect conductive wire antenna of length $l=100m$ and radius $a=0.1m$. The source is located at $0.1l$. The antenna operates at $f=100Hz$ and is placed in a conductive homogeneous medium with conductivity $\sigma_1=4S/m$. The result is obtained by using the built-in function quad with different filaments.

Fig. 3. A replica of [5,Fig.6], representing the theoretical current distribution at different frequencies for a metallic pipe of 40m immersed in seawater with conductivity $\sigma_1=4S/m$. The conductivity of the metal pipe is $\sigma_s=4.2 \cdot 10^6 S/m$.

Fig. 4. A replica of [5,Fig.4], representing the theoretical current distribution along a drill string of 3600m (radius 10cm) through a homogeneous medium with $\sigma_1=0.5S/m$. The conductivity of the drill string is $\sigma_s=2 \cdot 10^6 S/m$.

For the simulations in an inhomogeneous medium, the simple model of Fig. 6 is considered. In this model, the wire antenna was assumed to have a length $l=1000m$, the conductivities of layer 1 and layer 3 are identical and equal to $\sigma_1=\sigma_3=1S/m$, whereas the conductivity of layer 2 is equal to $\sigma_2=4S/m$. The source is located at $z_s=0.1l$ and the interfaces $d_1=0.7l$ and $d_2=0.8l$, and the antenna is assumed to be a perfect conductor.

The theoretical current distribution using the quad function is presented in Fig. 7. It is observed that attenuation increases both when the frequency is increased and when the conductivity increases in layer 2. In Fig. 8, $d_1=0.6l$ and $d_2=0.8l$, the result at 10Hz is compared to the attenuation of a TEM wave in a similar environment. It is observed that the current along the wire antenna attenuates faster than the TEM wave, which proves the conclusion of Wait in [6] that a signal along the antenna will attenuate at least as great as that of plane waves in a conductive medium.
4. Conclusion and outlook

A theoretical approach to obtain the current distribution along a wire antenna placed in an inhomogeneous layered medium is presented. The numerical results show that the attenuation will vary according to the conductivity of the environment and at least as fast as a plane wave propagating in a similar environment.

References